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Discrete Mathematics 174 (1997) 337–345

DISCRETE MATHEMATICS

Projective planes and dihedral groups

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Received 8 November 1994; revised 14 August 1995

Abstract

In the present article we shall show that any two disjoint Baer subplanes of $\mathrm{PG}(2,q^2)$ are contained in exactly one Singer–Baer partition.

Given two disjoint Baer subplanes of $P = \mathrm{PG}(2,q^2)$ with Baer involutions τ_0 and τ_1 we shall see that $\delta := \tau_0\tau_1$ is a projective collineation whose order is a divisor of $q^2 - q + 1$. If $o(\delta) = q^2 - q + 1$, then the point orbits of P under the action of $\langle\delta\rangle$ are so-called Kestenband arcs.

1. Introduction

Let $P = \mathrm{PG}(d,q^2)$ be a finite desarguesian projective space of square order q^2. A *Baer subspace* B of P is a d-dimensional projective subspace of P of order q. Given a Baer subspace B of P there exists exactly one collineation τ of P with the property that the fixed points of τ are exactly the points of B. This collineation τ is of order 2 and is called the *Baer involution* defined by B. The study of the following questions has been proposed to the author by Bill Kantor [12].

Let B_0 and B_1 be two Baer subspaces of P, and let τ_0 and τ_1 be the corresponding Baer involutions. Then $D := \langle\tau_0,\tau_1\rangle$ is a dihedral group.

What are the possible orders of D?

Which geometric structure is induced by D in P?

In the present paper we shall study the case where B_0 and B_1 are two disjoint Baer subplanes of the desarguesian projective plane $P = \mathrm{PG}(2,q^2)$. Our main result is as follows (for the relevant definitions see Section 2):

Theorem 1.1. *Let B_0 and B_1 be two disjoint Baer subplanes of the desarguesian projective plane $P = \mathrm{PG}(2,q^2)$, and let τ_0 and τ_1 be the Baer involutions of B_0 and B_1, respectively. Let $\delta := \tau_0\tau_1$, and let r be the order of δ.*

(a) *The order of δ is a divisor of $q^2 - q + 1$.*

(b) *There is a Singer cycle σ of P and a natural number n such that $\delta = \sigma^n$.*

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PII S0012-365X(96)00347-0

(c) *The set*

$$\mathscr{P}(B_0,B_1) := \{B_0, \delta(B_0), \ldots, \delta^{r-1}(B_0)\}$$

is a set of mutually disjoint Baer subplanes. If $r = q^2-q+1$, *then* $\mathscr{P}(B_0,B_1)$ *is a linear Baer partition containing* B_0 *and* B_1. *Furthermore, it follows that* $\mathscr{P}(B_0,B_1) = \mathscr{P}(\sigma)$.

(d) *The group* $D := \langle \tau_0, \tau_1 \rangle$ *is a dihedral group of order* $2r$. *The reflections of* D *are exactly the Baer involutions of the Baer subplanes contained in* $\mathscr{P}(B_0,B_1)$.

(e) *If* $r = q^2 - q + 1$, *then the point orbits of* P *under the action of* $\langle \delta \rangle$ *are the complete* $(q^2 - q + 1)$*-arcs introduced by Kestenband* [14].

(f) *There exist two disjoint Baer subplanes* C_0 *and* C_1 *with Baer involutions* γ_0 *and* γ_1 *such that the element* $\gamma_0\gamma_1$ *is of order* $q^2 - q + 1$ *and* $B_0, B_1 \in \mathscr{P}(C_0, C_1)$. *The partition* $\mathscr{P}(C_0, C_1)$ *is uniquely determined by* B_0 *and* B_1.

The present paper is organized as follows: In Section 2 we provide the necessary information about Baer subplanes, Singer cycles and Kestenband arcs. Theorem 1.1 is proved in Section 3. At the end of this paper we discuss some related problems and open questions.

2. Baer partitions and Kestenband arcs

Proposition 2.1. *Let* B_0 *and* B_1 *be two Baer subplanes of the desarguesian projective plane* $P = \mathrm{PG}(2,q^2)$, *and let* τ_0 *and* τ_1 *be the Baer involutions defined by* B_0 *and* B_1, *respectively.*

a) *Let* α *be a collineation of* P. *Then we have* $\alpha(B_0) = B_1$ *if and only if* $\alpha^{-1}\tau_1\alpha = \tau_0$.

b) *Any two Baer involutions are conjugate under some projective collineation of* P.

Proof. (a) Let $\alpha(B_0) = B_1$, and let $\tau := \alpha^{-1}\tau_1\alpha$. Then τ is of order 2, and for any point x of B_0 we have

$$\tau(x) = (\alpha^{-1}\tau_1\alpha)(x) = \alpha^{-1}(\tau_1(\alpha(x))) = \alpha^{-1}(\alpha(x)) = x,$$

since $\alpha(x) \in B_1$. Hence τ is an involution fixing all points of B_0, that is, τ is the Baer involution defined by B_0, that is, $\tau = \tau_0$.

Conversely, suppose that $\alpha^{-1}\tau_1\alpha = \tau_0$, that is, $\tau_1\alpha = \alpha\tau_0$. Let x be a point of B_0. In order to show that $\alpha(B_0) = B_1$ we have to show that $\tau_1(\alpha(x)) = \alpha(x)$. This last equation follows from

$$(\tau_1\alpha)(x) = (\alpha\tau_0)(x) = \alpha(\tau_0(x)) = \alpha(x).$$

(b) Let B_0 and B_1 be two Baer subplanes of P, and let Q_0 and Q_1 be two quadrangles of B_0 and B_1, respectively (a quadrangle is a set of four points, no three of them collinear). Since the group $\mathrm{PGL}_3(q^2)$ acts transitively on the quadrangles of P (see [8]), it follows that there exists a projective collineation α with $\alpha(Q_0) = Q_1$. Since there exists exactly one Baer subplane containing Q_1 (see [5]), it follows that $\alpha(B_0) = B_1$. By a), the Baer involutions defined by B_0 and B_1 are conjugate under α. □

Proposition 2.2. *Let $P = \mathrm{PG}(2, q^2)$, and let B_0 and B_1 be two Baer subplanes with Baer involutions τ_0 and τ_1. Let $\delta := \tau_0\tau_1$.*

(a) *The collineation δ is a projective collineation.*

(b) *The Baer subplanes B_0 and B_1 are disjoint if and only if δ has no fixed points.*

(c) *Let D be an element of $\mathrm{GL}_3(q^2)$ inducing δ. If B_0 and B_1 are disjoint, then the characteristic polynomial f_D of D is irreducible.*

Proof. (a) Let γ be a projective collineation with $B_1 = \gamma(B_0)$. Then, by Proposition 2.1, $\tau_0 = \gamma^{-1}\tau_1\gamma$, and it follows that

$$\delta = \tau_0\tau_1 = \gamma^{-1}\tau_1\gamma\tau_1 = \gamma^{-1}(\tau_1^{-1}\gamma\tau_1).$$

Since the group of all projective collineations is a normal subgroup of the group of all collineations of P, it follows that δ is a projective collineation.

(b) Suppose that B_0 and B_1 have a common point x. Then it follows that $\delta(x) = \tau_0\tau_1(x) = x$.

Conversely, let $\delta(x) = x$ for some point x of P. Then $x = \delta(x) = \tau_0\tau_1(x)$, hence $\tau_0(x) = \tau_1(x)$. If $\tau_0(x) = \tau_1(x) = x$, then x is a common point of B_0 and B_1. If $\tau_0(x) \neq x$, then the line through x and $\tau_0(x)$ is fixed by τ_0 and τ_1, that is, B_0 and B_1 have a common line. Then B_0 and B_1 have also a point in common (see [2] or [20]).

(c) Assume that f_D is reducible. Then f_D admits a factor of degree 1 implying that D has an eigenvector. Thus δ has a fixed point, a contradiction to (b). □

Let P be a projective plane of order q. A *Singer cycle* of P is a collineation of P of order $q^2 + q + 1$ permuting all points of P in a single cycle. If P is a finite desarguesian projective plane, then P always admits a Singer cycle (see [17] or [11], Theorem 4.2.1). The following proposition is well known (see [17] or [11], Theorem 4.3.5).

Proposition 2.3. *Let σ be a Singer cycle of a desarguesian projective plane of order q^2. Then the point orbits of P under the action of the group $\langle\sigma^{q^2-q+1}\rangle$ form a partition of P into $q^2 - q + 1$ disjoint Baer subplanes.*

Given a Singer cycle σ of a desarguesian projective plane P of order q^2 we call the partition of P into Baer subplanes described in Proposition 2.3 the *linear Baer partition induced by σ*, and we denote it by $\mathscr{P}(\sigma)$.

Proposition 2.4. *Let P be a desarguesian projective plane of order q^2, and let σ_1 and σ_2 be two Singer cycles of P with $\langle\sigma_1\rangle = \langle\sigma_2\rangle$. Then $\mathscr{P}(\sigma_1) = \mathscr{P}(\sigma_2)$.*

Proof. The cyclic group $\langle\sigma_1\rangle = \langle\sigma_2\rangle$ has exactly one subgroup of order q^2+q+1 with the consequence that $\langle\sigma_1^{q^2-q+1}\rangle = \langle\sigma_2^{q^2-q+1}\rangle$. Therefore σ_1 and σ_2 induce the same linear Baer partition. □

Proposition 2.5. *Let σ be a Singer cycle of the projective plane $P = \mathrm{PG}(2,q^2)$, and let $\mathscr{P}(\sigma)$ be the linear Baer partition induced by σ. Let B be a Baer subplane of P with Baer involution τ. Then we have $B \in \mathscr{P}(\sigma)$ if and only if $\tau\sigma^{q^2-q+1} = \sigma^{q^2-q+1}\tau$.*

Proof. By Proposition 2.3, the orbits of P under the action of the group $\langle\sigma^{q^2-q+1}\rangle$ are the point sets of the Baer subplanes of $\mathscr{P}(\sigma)$.

Suppose that $B \in \mathscr{P}(\sigma)$. Then $\sigma^{q^2-q+1}(B) = B$. By Proposition 2.1, it follows that

$$(\sigma^{q^2-q+1})^{-1}\tau\sigma^{q^2-q+1} = \tau,$$

i.e., $\tau\sigma^{q^2-q+1} = \sigma^{q^2-q+1}\tau$.

Conversely, suppose that $\tau\sigma^{q^2-q+1} = \sigma^{q^2-q+1}\tau$. Let x be a point of B. Then we have

$$\tau(\sigma^{q^2-q+1}(x)) = \sigma^{q^2-q+1}(\tau(x)) = \sigma^{q^2-q+1}(x).$$

Hence $\sigma^{q^2-q+1}(x) \in B$. It follows that $\sigma^{q^2-q+1}(B) = B$ which implies that $B \in \mathscr{P}(\sigma)$. □

Next, we shall introduce the so-called Kestenband arcs.

Theorem 2.1 (Kestenband [14]). *Let U_0 and U_1 be the hermitian unitals in $\mathrm{PG}(2,q^2)$ defined by the equations $x^tIx^{(q)} = 0$ and $x^tHx^{(q)} = 0$, respectively. If the characteristic polynomial f_H of H is irreducible over $\mathrm{GF}(q^2)$, then the point set $U_0 \cap U_1$ is a (q^2-q+1)-arc.* □

In fact, in his paper Kestenband [14] determined the intersection of any two hermitian unitals.

The (q^2-q+1)-arcs introduced in Theorem 2.1 are called *Kestenband arcs*. Another description of the Kestenband arcs is due to Fisher, Hirschfeld and Thas [10] and to Boros and Szönyi [1] who proved independently the following theorem:

Theorem 2.2. *Let $P = \mathrm{PG}(2,q^2)$, and let σ be a Singer cycle of P. Furthermore let p be an arbitrary point of P, and let $b := q^2+q+1$ and $k := q^2-q+1$.*

(a) *Given $i \in \{0,1,\ldots,q^4+q^2\}$ there exist unique integers $r \in \{0,1,\ldots,b-1\}$ and $s \in \{0,1,\ldots,k-1\}$ such that $i \equiv r \bmod b$ and $i \equiv s \bmod k$. Hence any point $\sigma^i(p)$ of P can be identified with such a pair (r,s).*

(b) *For $s = 0,1,\ldots,k-1$ let*

$$A_s := \{(r,s) \mid r = 0,\ldots,b-1\}.$$

Then A_s is a Baer subplane, and the family $\{A_0,A_1,\ldots,A_{k-1}\}$ is the linear Baer partition $\mathscr{P}(\sigma)$ induced by σ.

(c) *For* $r = 0, 1, \ldots, b-1$ *let* $K_r := \{(r,s) \mid s = 0, \ldots, k-1\}$. *Then the sets* $K_0, K_1, \ldots, K_{b-1}$ *are mutually disjoint Kestenband arcs.*

(d) *For* $q > 2$, *Kestenband arcs are complete* $(q^2 - q + 1)$*-arcs.*

The completeness of the Kestenband arcs for $q > 2$ is of particular interest because of the following theorem due to Segre [16]: [1]

Theorem 2.3. *A complete k-arc in* PG$(2,q)$ *with q even is either a hyperoval, that is,* $k = q + 2$, *or we have* $k \leqslant q - \sqrt{q} + 1$.

The completeness of the Kestenband arcs shows that the estimate of Segre is sharp for desarguesian projective planes of even square order q^2 with $q > 2$. Meanwhile there exist several proofs for the completeness of the Kestenband arcs: Besides the above mentioned papers of Fisher, Hirschfeld and Thas and of Boros and Szőnyi there is a paper of Kestenband [15], Ebert [9] and an article of Cossidente [6] about the completeness of Kestenband arcs. Additional information about these arcs or generalizations of these arcs can be found in [7]. A survey on arcs is included in [19].

3. Disjoint Baer subplanes

In this section we shall prove Theorem 1.1. As a first step we shall show that any two disjoint Baer subplanes of $P = \mathrm{PG}(2, q^2)$ are contained in exactly one linear Baer partition.

Theorem 3.1. *Let* B_0 *and* B_1 *be two disjoint Baer subplanes of the desarguesian projective plane* $P = \mathrm{PG}(2, q^2)$. *Then there exists exactly one linear Baer partition* $\mathscr{P}(\sigma)$ *induced by a Singer cycle* σ *of P containing* B_0 *and* B_1.

Proof. *Step* 1: *There exists a linear Baer partition* $\mathscr{P}(\sigma)$ *induced by a Singer cycle* σ *of P containing* B_0 *and* B_1.

For, denote by F the field GF(q^2). Let τ_0 and τ_1 be the Baer involutions defined by B_0 and B_1, respectively. Then we have $\tau_0, \tau_1 \in \mathrm{P}\Gamma\mathrm{L}_3(q^2)$, and by Proposition 2.2, it follows that $\delta := \tau_0\tau_1$ is a projective collineation, that is, $\delta \in \mathrm{PGL}_3(q^2)$.

Since τ_0 and τ_1 are also elements of the group $\Gamma\mathrm{L}_3(q^2)$, the element $D := \tau_0\tau_1$ is an element of $\Gamma\mathrm{L}_3(q^2)$. As in the proof of Proposition 2.2 it is easily shown that D is even an element of $\mathrm{GL}_3(q^2)$. The projective collineation induced by D is δ. By Proposition 2.2, the characteristic polynomial f_D of D is irreducible. Hence the set

$$\mathscr{F} := \{p(D) \mid p \in F[x]\} \subseteq \mathrm{GL}_3(q^2) \subseteq \Gamma\mathrm{L}_3(q^2)$$

is a field isomorphic to GF(q^6). Let S be a primitive element of $\mathscr{F}$, and let σ be the projective collineation of P induced by S. Then σ is a Singer cycle (see Theorem 4.2.1

[1] A 3-arc is never complete.

of [11]). Let $\mathscr{P}(\sigma)$ be the linear Baer partition induced by σ. It remains to show that B_0 and B_1 are elements of $\mathscr{P}(\sigma)$. For, we consider the application

$$T : \mathrm{GL}_3(q^2) \to \mathrm{GL}_3(q^2)$$
$$T : A \mapsto \tau_0^{-1} A \tau_0 = \tau_0 A \tau_0.$$

Since $D = \tau_0\tau_1$, we have $\tau_0 D \tau_0 = \tau_0^2 \tau_1 \tau_0 = \tau_1\tau_0 = D^{-1}$. Let $p \in F[x]$. Then $p(D) \in \mathscr{F}$, and we have

$$\tau_0\, p(D) \tau_0 = p(D^{-1}) \in \mathscr{F}.$$

It follows that $\mathscr{F}$ is invariant under T. Because of

$$T(p_1(D) + p_2(D)) = p_1(D^{-1}) + p_2(D^{-1}) = T(p_1(D)) + T(p_2(D)),$$

and

$$T(p_1(D) \cdot p_2(D)) = p_1(D^{-1}) \cdot p_2(D^{-1}) = T(p_1(D)) \cdot T(p_2(D))$$

it follows that T restricted to $\mathscr{F}$ is a (field) automorphism of $\mathscr{F}$. Since $T|_{\mathscr{F}}$ is of order 2, it follows that

$$T|_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}, \qquad T|_{\mathscr{F}} : A \mapsto A^{q^3}.$$

In particular, we have $\tau_0 A \tau_0 = A^{q^3}$ for all $A \in \mathscr{F}$. Since $S \in \mathscr{F}$, it follows that $\tau_0 S \tau_0 = S^{q^3}$. Hence $\tau_0 \sigma \tau_0 = \sigma^{q^3}$. We are now able to show that B_0 is an element of $\mathscr{P}(\sigma)$: The Baer subplanes contained in $\mathscr{P}(\sigma)$ are the orbits of P under the action of σ^{q^2-q+1}. Because of

$$\begin{aligned}
\tau_0 \sigma^{q^2-q+1} \tau_0 &= (\tau_0 \sigma \tau_0)^{q^2-q+1} = (\sigma^{q^3})^{q^2-q+1} \\
&= (\sigma^{1+q^3-1})^{q^2-q+1} = \sigma^{q^2-q+1} \sigma^{(q-1)(q^2+q+1)(q^2-q+1)} \\
&= \sigma^{q^2-q+1} (\sigma^{q^4+q^2+1})^{q-1} \\
&= \sigma^{q^2-q+1}
\end{aligned}$$

it follows that $\tau_0 \sigma^{q^2-q+1} = \sigma^{q^2-q+1} \tau_0$. By Proposition 2.5, we have $B_0 \in \mathscr{P}(\sigma)$. The same reasoning shows that $B_1 \in \mathscr{P}(\sigma)$.

Step 2: *It remains to show that there exists exactly one linear Baer partition $\mathscr{P}(\sigma)$ induced by a Singer cycle σ containing B_0 and B_1.*

If σ is a Singer cycle of P, then the group $\langle\sigma\rangle$ is a cyclic group of order q^4+q^2+1 containing exactly $\phi(q^4+q^2+1)$ generators, where ϕ denotes the Euler function. Hence the group $\langle\sigma\rangle$ contains exactly $\phi(q^4+q^2+1)$ Singer cycles. We shall use the following facts about a desarguesian projective plane of square order q^2:

- The plane P contains exactly $q^3(q^{3+1})(q^2+1)$ Baer subplanes (see [11, Corollary 3 of Lemma 4.3.1]).
- Given a Baer subplane B of P there are exactly $\frac{1}{3}q^4(q^2-1)(q-1)^2$ Baer subplanes of P disjoint to B (see [2] or [18]).

- The plane P admits exactly $\frac{1}{3}q^6(q^4-1)(q^2-1)\phi(q^4+q^2+1)$ Singer cycles (see [11, Corollary 3 of Theorem 4.2.1]).

Let us consider the set $\mathscr{S}$ of all triples (B,B',σ) such that B and B' are two disjoint Baer subplanes of P contained in the linear Baer partition $\mathscr{P}(\sigma)$ induced by the Singer cycle σ. If σ is a Singer cycle, then the linear Baer partition $\mathscr{P}(\sigma)$ contains exactly $k := q^2-q+1$ mutually disjoint Baer subplanes. If we denote by s the number of Singer cycles of P, then it follows that

$$\begin{aligned}|\mathscr{S}| &= sk(k-1)\\ &= \tfrac{1}{3}q^6(q^4-1)(q^2-1)\phi(q^4+q^2+1)(q^2-q+1)q(q-1) =: s_1.\end{aligned}$$

On the other hand, it follows from part (a) that for any two disjoint Baer subplanes B and B' there exists a Singer cycle σ such that $B,B' \in \mathscr{P}(\sigma)$. By Proposition 2.4, there are $\phi(q^4+q^2+1)$ Singer cycles α in $\langle\sigma\rangle$ each of them fulfilling the equation $\mathscr{P}(\sigma) = \mathscr{P}(\alpha)$.

Therefore if we denote by b and by b_0 the number of Baer subplanes of P and the number of Baer subplanes of P disjoint to a given Baer subplane, respectively, then we get

$$\begin{aligned}|\mathscr{S}| &\geqslant bb_0\phi(q^4+q^2+1)\\ &= q^3(q^3+1)(q^2+1)\tfrac{1}{3}q^4(q^2-1)(q-1)^2\phi(q^4+q^2+1) =: s_2.\end{aligned}$$

Since $s_1 = s_2$, it follows that any two disjoint Baer subplanes B and B' are contained in exactly one linear Baer partition $\mathscr{P}(\sigma)$ for some Singer cycle σ. $\square$

We are now able to prove Theorem 1.1.

Proof. Let B_0 and B_1 be two disjoint Baer subplanes of $P = \mathrm{PG}(2,q^2)$, and let τ_0 and τ_1 be the Baer involutions of B_0 and B_1, respectively. Let $\delta := \tau_0\tau_1$. By Theorem 3.1, there exists a Singer cycle σ such that B_0 and B_1 are contained in the linear Baer partition $\mathscr{P}(\sigma)$ induced by σ.

Let $b := q^2+q+1$ and $k := q^2-q+1$. Let p be a point of P. Since σ is a Singer cycle, any point of P is of the form $\sigma^i(p)$ for some $i \in \{0,1,\dots,q^4+q^2\}$. By Theorem 2.2, the point $\sigma^i(p)$ can be written in the form (r,s), where $r \in \{0,1,\dots,b-1\}$ and $s \in \{0,1,\dots,k-1\}$ and $i \equiv r \bmod b$ and $i \equiv s \bmod k$. Furthermore, for $s = 0,\dots,k-1$ the set $A_s := \{(r,s) \mid r = 0,1,\dots,b-1\}$ is a Baer subplane, and the family $\{A_0,A_1,\dots,A_{k-1}\}$ is the linear Baer partition induced by σ. For $j = \{0,1,\dots,k-1\}$ let α_j be the Baer involution of the Baer subplane A_j.

Step 1: *Let $\beta := \alpha_1\alpha_0$. Then β is given by the application $\beta(r,s) = (r,s+2)$, and it exists an integer m such that $\beta = \sigma^m$. Furthermore, β is of order $k = q^2-q+1$, and $G := \langle\alpha_0,\alpha_1\rangle$ is a dihedral group of order $2(q^2-q+1)$.* If $i \equiv r \bmod b$ and $i \equiv s \bmod k$, then $i+1 \equiv r+1 \bmod b$ and $i+1 \equiv s+1 \bmod k$. Hence $\sigma(r,s) = (r+1,s+1)$ for all points (r,s), where $r+1$ and $s+1$ are taken modulo b and k, respectively. Using Hall's multiplier theorem it is shown in [10, Lemma 3.1] that the Baer involution α_0 of A_0 is given by $\alpha_0(r,s) = (r,k-s)$. Since $\sigma(A_0) = A_1$, we have $\alpha_1 = \sigma\alpha_0\sigma^{-1}$, and it follows that $\alpha_1(r,s) = (r, k-s+2)$ for all points (r,s). As a consequence we have $\beta(r,s) = \alpha_1\alpha_0(r,s) = (r,s+2)$. Since 2 and $q^2-q+1 = q(q-1)+1$ are relatively prime, β is of order q^2-q+1. Since $\sigma(r,s) = (r+1,s+1)$ and $\beta(r,s) = (r,s+2)$, there exists an integer m with $\beta = \sigma^m$.

Step 2: *We have* $\{A_0, A_1, \ldots, A_{k-1}\} = \{A_0, \beta(A_0), \ldots, \beta^{k-1}(A_0)\}$.

From $\beta(r,s) = (r,s+2)$ it follows that $\beta^j(A_0) = A_{2j}$, where $2j$ is taken modulo k. Since 2 and $k = q^2-q+1$ are relatively prime, we obtain

$$\{A_0, A_1, \ldots, A_{k-1}\} = \{A_0, \beta(A_0), \ldots, \beta^{k-1}(A_0)\}.$$

Step 3: *The Baer involutions* $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ *are the reflections of the dihedral group* $G := \langle \alpha_0, \alpha_1 \rangle$.

For, let $j \in \{0, 1, \ldots, k-1\}$. By Step 2, there exists an integer i such that $A_j = \beta^i(A_0)$. Hence $\alpha_j = \beta^i \alpha_0 (\beta^i)^{-1}$. It follows that α_j is an element of G conjugate to α_0, that is, α_j is a reflection of G.

Step 4: *For each point* x *of* P *the orbit of* x *under the action of* $\langle \beta \rangle$ *is a Kestenband arc.*

Since x is of the form (r,s) for some $r \in \{0, 1, \ldots, b-1\}$ and for some $s \in \{0, 1, \ldots, k-1\}$, it follows from $\beta(r,s) = (r,s+2)$ that the orbit of x under $\langle \beta \rangle$ is the set $K_r = \{(r,s) \mid s = 0, \ldots, k-1\}$, that is, a Kestenband arc.

Step 5: *The Baer involutions* τ_0 *and* τ_1 *are reflections of the dihedral group* $G = \langle \alpha_0, \alpha_1 \rangle$. *In particular there exists an integer* n *such that* $\delta = \beta^n$, *where* $\delta = \tau_0\tau_1$.

Since $\{A_0, A_1, \ldots, A_{k-1}\}$ is the linear Baer partition $\mathscr{P}(\sigma)$ induced by σ, we have $B_0, B_1 \in \{A_0, A_1, \ldots, A_{k-1}\}$. By Step 3, τ_0 and τ_1 are reflections of the dihedral group G. In particular there exists an integer n such that $\delta = \beta^n$.

(a) Since $\delta = \beta^n$ for some integer n and since β is of order q^2-q+1, the order of δ is a divisor of q^2-q+1.

(b) Since $\delta = \beta^n$ for some integer n (Step 5) and $\beta = \sigma^m$ for some integer m (Step 1), it follows that $\delta = \sigma^{mn}$.

(c) By Step 5, we have

$$\mathscr{P}(B_0, B_1) \subseteq \{A_0, A_1, \ldots, A_{k-1}\} = \mathscr{P}(\sigma).$$

Therefore $\mathscr{P}(B_0, B_1)$ is a set of r mutually disjoint Baer subplanes with the property that $\mathscr{P}(B_0, B_1)$ is a linear Baer partition if $r = q^2-q+1$.

(d) By Step 5, $D = \langle \tau_0, \tau_1 \rangle$ is a dihedral group contained in the group $G = \langle \alpha_0, \alpha_1 \rangle$. The reflections of D are exactly the Baer involutions of the Baer subplanes $B_0, \delta(B_0), \ldots, \delta^{r-1}(B_0)$, that is, of the Baer subplanes contained in $\mathscr{P}(B_0, B_1)$.

(e) If δ is of order q^2-q+1, then $\langle \delta \rangle = \langle \beta \rangle$. By Step 4, the point orbits of P under the action of $\langle \beta \rangle$ are Kestenband arcs.

(f) Since $B_0, B_1 \in \{A_0, A_1, \ldots, A_{k-1}\}$, we can take $C_0 := A_0$ and $C_1 := A_1$. □

We conclude this paper with the following remarks and open questions:

1. If $P = \mathrm{PG}(1, q^2)$, then the Baer subspaces of P are the circles of the inversive plane I defined by P. The Baer involutions are the inversions of I. Let τ_0 and τ_1

be two inversions of I. The structure of the dihedral group $\langle \tau_0, \tau_1 \rangle$ has been analysed in [4].

2. Let $P = \mathrm{PG}(2,F)$ be an infinite projective plane over some field F. If F is not countable, then P does not admit a Singer cycle. If τ_0 and τ_1 are the Baer involutions of two disjoint Baer subplanes, then the following problem arises: Does there exist a group G acting regularly on the point set of P such that G is the direct product of two subgroups U and W with the property that the orbits of P under the action of U are Baer subplanes containing B_1 and B_2 and the orbits of P under W are maximal arcs? This problem is studied by T. Meixner and the author.

3. Let $P = \mathrm{PG}(2,q^2)$, and let $\mathscr{P}$ be a partition of the point set of P into Baer subplanes. Then the following question arises: Is $\mathscr{P}$ necessarily a *linear* Baer partition?

In $P = \mathrm{PG}(2,9)$, Meixner and the author have found with the aid of Cayley four Baer involutions with pairwise disjoint Baer subplanes such that no three of them are contained in a linear Baer partition.

Acknowledgements

I wish to thank Albrecht Beutelspacher, Jörg Eisfeld, Chris Fisher, Udo Heim, and Thomas Meixner for many valuable discussions.

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